THEORY OF EARLY ONSET OF TURBULENCE IN A DISPERSION

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The effects of phase interaction are considered for the hydrodynamic instability of a flowing dispersion, and a model is proposed for premature onset of turbulence.

A disperse phase can stabilize the laminar flow of a suspension or aerosol by impeding the growth of neutral perturbations and the onset of turbulent conditions. This stabilizing action is seen also in the shift of the curves for neutral stability towards higher Reynolds number R [1, 2] and in the occurrence of additional dissipation of pulsation energy as a result of phase interaction [3, 4].

These conclusions from the theory are closely confirmed by tests on the motion of finely divided or nearly equal-density systems, where turbulence starts at R_* higher than those for the homogeneous dispersion medium. However, systems with coarser particles and substantial density differences show the converse effect (lower R_*). This has been observed [5] for a liquid containing sand in vertical pipes, where R_* was only 150-200 instead of 2300. Similar results have been obtained with oil in sands [6]. Loss of laminar flow at low R_* is characteristic also of emulsions having disperse material less dense than the dispersion medium, e.g., water-oil emulsions of very low concentration [7]. The quantity R_* is not an invariant characteristic of the type of motion but is dependent on the physical parameters of the phases and the concentration.

This indicates that a different mechanism accompanies the usual mechanism for loss of hydrodynamic stability (from inertial effects, undamped shear waves, and secondary flows); energy is transferred to pulsations and causes premature onset of turbulence. This energy transfer must be due to interaction between phases, which produces differences between the local speeds of liquid and particles; it does not require shear in order to arise. A preliminary analysis of early turbulence thus can be based on the stability of a one-dimensional homogeneous two-phase flow.

§1. The motion is described via a model with two interpenetrating continuous media [3], which is supplemented by an expression for the phase-interaction forces implied by [8]. This expression takes account of viscous resistance and buoyancy, as well as forces due to the excess pressure gradient and the acceleration of the liquid in transient motion of the particles. These forces are especially important when the two media are nearly equal in density. We assume that the force of viscous interaction is linear in the relative velocity of the phases and can be represented as a product of the Stokes force and a definite function of the Archimedes number A and the volume concentration ρ.

The equations for conservation of the mass and momentum of the phases are

$$-\frac{\partial \rho}{\partial t}+\operatorname{div}[(1-\rho)\mathbf{v}]=0, \qquad \frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{w})=0,$$

$$\left\{[\rho+\varkappa(1-\rho)+\xi\varkappa\rho]\frac{\partial}{\partial t}+\varkappa(1-\rho)(\mathbf{v}\nabla)+\right.$$

$$\left.+(1+\xi\varkappa)\rho(\mathbf{w}\nabla)\right\}\mathbf{v}-\xi\varkappa\rho\left[\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right]\mathbf{w}=$$

$$=-\nabla\mathbf{p}+\nabla\boldsymbol{\tau}^{(1)}+\varkappa\mathbf{g}-\beta\rho K(\mathbf{v}-\mathbf{w}),$$

$$\left[(1+\xi\varkappa)\frac{\partial}{\partial t}+(\mathbf{w}\nabla)+\xi\varkappa(\mathbf{v}\nabla)\right]\mathbf{w}-$$

$$-(1+\xi\varkappa)\left[\frac{\partial}{\partial t}+(\mathbf{w}\nabla)\right]\mathbf{v}=$$

$$=-\frac{1}{\rho}\nabla\mathbf{q}+\frac{1}{\rho}\nabla\boldsymbol{\tau}^{(2)}+(1-\varkappa)\mathbf{g}+\beta K(\mathbf{v}-\mathbf{w}),$$

$$\beta=\frac{9\mu_0}{2a^2d_2}, \qquad \varkappa=\frac{d_1}{d_2}, \qquad K=K(\rho, A). \tag{1.1}$$

Here w and v are the velocities of the solid and liquid phases; $\mathbf{p}, \boldsymbol{\tau}^{(1)}$ and $\mathbf{q}, \boldsymbol{\tau}^{(2)}$ are the tensors for the pressure and viscous stresses in the liquid and dispersed phases, as referred to the density d_2 of the particles; d_1 is the density of the liquid; μ_0 is the viscosity of the liquid; a is the radius of a (spherical a particle; g is the acceleration due to gravity; $K(\rho, A)$ is a function that takes account of the increase in the effective viscous force in hindered flow around the particles; and $\xi = \xi(\rho)$ is the adjoint-mass coefficient $\xi \leq 1/2$). We assume that $\tau^{(1)}$ and $\tau^{(2)}$ are expressed in terms of the velocity gradients in the manner usual for the hydrodynamics of a viscous liquid.

A one-dimensional uniform flow is described by the following relations (the axis $x = x_1$ is opposite in sense to the vector g):

$$w_i=0, \quad v_i=u\delta_{i1}, \quad p=\text{const}-(\varkappa g-\beta\rho K u)x,$$

$$q=\text{const}, \qquad u=\frac{(1-\varkappa)g}{\beta K}=\frac{2}{9}\frac{a^2(d_2-d_1)g}{\mu_0 K}. \tag{1.2}$$

Equations (1.1) take the following form when linearized with respect to small perturbations in the quantities (denoted by asterisks) relative to the steady state of (1.2):

$$\left\{[\rho+\varkappa(1-\rho)+\xi\varkappa\rho]\frac{\partial}{\partial t}+\varkappa(1-\rho)u\frac{\partial}{\partial x}\right\}v_i^*-$$

$$-\xi\varkappa\rho\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)w_i^*=-$$

$$-\nabla p^*+\mu^{(1)}\triangle v_i^*+\gamma^{(1)}\nabla(\operatorname{div}\mathbf{v}^*)-$$

$$-\beta\rho K u_i^*-\beta\left(K+\rho\frac{\partial K}{\partial\rho}\right)u\delta_{1i}\rho^*,$$

$$\gamma^{(i)}=\frac{1}{3}\mu^{(i)}+\zeta^{(i)}$$

$$\left[(1+\xi\varkappa)\frac{\partial}{\partial t}+\xi\varkappa u\frac{\partial}{\partial x}\right]w_i^*-(1+\xi\varkappa)\frac{\partial v_i^*}{\partial t}=$$

$$=-\frac{1}{\rho}\frac{\partial g^{(i)}}{\partial\rho}\frac{\partial\rho^*}{\partial x_i}+\frac{\mu^{(2)}}{\rho}\triangle w_i^*+$$

$$+\frac{\gamma^{(2)}}{\rho}\nabla(\operatorname{div}\mathbf{w}^*)+\beta K u_i^*+\beta\frac{\partial K}{\partial\rho}\delta_{1i}u\rho^*,$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\rho^*=(1-\rho)\operatorname{div}\mathbf{v}^*,$$

$$\frac{\partial\rho^*}{\partial t}=-\rho\operatorname{div}\mathbf{w}^*, \quad \mathbf{u}^*=\mathbf{v}^*-\mathbf{w}^* \tag{1.3}$$

The pressure tensor for the liquid is assumed to be spherical, while the equilibrium normal stresses of the solid vary with direction ($q^{(1)} \neq q^{(2)} = q^{(3)}$), as follows from [9, 10]. Here $\mu^{(i)}$ and $\zeta^{(i)}$ are the shear and bulk viscosities of the phases respectively, as referred to d_2. We put

$$\{p^*, \rho^*, \mathbf{v}^*, \mathbf{w}^*\} = \{P, R, \mathbf{V}, \mathbf{W}\} e^{i(\omega t + \mathbf{kx})}.$$

These relations are substituted into (1.3), and V and W are eliminated from the system of algebraic equations to give

$$ik^2P + (r_1k_1 + s_1k_2)R = 0, \quad (s_2k_2 + r_2k_1)R = 0,$$

$$r_1 = \gamma^{(1)}k_1\frac{\omega + uk_1}{1-\rho} + \frac{1}{k_1}\left(\frac{\omega + uk_1}{1-\rho}a_1 + \frac{\omega}{\rho}b_1\right) +$$

$$+ \beta\left(K + \rho\frac{\partial K}{\partial \rho}\right)u,$$

$$r_2 = \frac{\gamma^{(2)}}{\rho}k_2\frac{\omega}{\rho} + \frac{1}{k_1}\left(\frac{\omega + uk_1}{1-\rho}a_2 + \frac{\omega}{\rho}b_2\right) +$$

$$+ \beta\frac{\partial K}{\partial \rho}u - \frac{ik_1}{\rho}\frac{\partial q^{(1)}}{\partial \rho},$$

$$s_1 = \gamma^{(1)}k_2\frac{\omega + uk_1}{1-\rho}, \quad s_2 = \frac{\gamma^{(2)}}{\rho}k_2\frac{\omega}{\rho} - i\frac{k_2}{\rho}\frac{\partial q^{(2)}}{\partial \rho},$$

$$a_1 = \mu^{(1)}k^2 + \beta\rho K +$$

$$+ i\{[\rho + \varkappa(1-\rho) + \xi\varkappa\rho]\omega + \varkappa(1-\rho)uk_1\},$$

$$a_2 = \beta K + i(1 + \xi\varkappa)\omega, \quad b_1 = \beta\rho K + i\xi\varkappa\rho(\omega + uk_1),$$

$$b_2 = \frac{\mu^{(2)}}{\rho}k^2 + \beta K + i[(1 + \xi\varkappa)\omega + \xi\varkappa uk_1]. \qquad (1.4)$$

The first equation in (1.4) defines P, while the second is the characteristic equation of the system and can be written as

$$(C_0 + iD_0)\omega^2 + (C_1 + iD_1)\omega + C_2 + iD_2 = 0,$$

$$C_0 = 0, \quad D_0 = 1 + \xi\varkappa,$$

$$C_1 = \beta K + \rho^{-1}(1-\rho)(\mu^{(2)} + \gamma^{(2)})k^2$$

$$D_1 = (\rho + \xi\varkappa)uk_1, \quad C_2 = \beta\rho\left[(1-\rho)\frac{\partial K}{\partial \rho} + K\right]u,$$

$$D_2 = -(1-\rho)\left(k_1^2\frac{\partial q^{(1)}}{\partial \rho} + k_2^2\frac{\partial q^{(2)}}{\partial \rho}\right). \qquad (1.5)$$

The stability conditions (absence of roots ω in (1.5) having negative imaginary parts) take the form

$$D_0C_1 > 0, \qquad (D_1C_1 - D_0C_2)C_2 - D_2C_1^2 > 0. \qquad (1.6)$$

The first condition in (1.6) is always met, while the second gives

$$\alpha_0k^4 + \alpha_1k^2 + \alpha_2 > 0$$

$$\alpha_0 = \frac{(1-\rho)^3}{\rho^2}(\mu^{(2)} + \gamma^{(2)})\left(k_1^2\frac{\partial q^{(1)}}{\partial \rho} + k_2^2\frac{\partial q^{(2)}}{\partial \rho}\right),$$

$$\alpha_1 = \beta(1-\rho)(\mu^{(2)} + \gamma^{(2)}) \times$$

$$\times\left\{k_1^2\left[2\frac{1-\rho}{\rho}K\frac{\partial q^{(1)}}{\partial \rho} + \right.\right.$$

$$\left.\left. + (\rho + \xi\varkappa)\left(K + (1-\rho)\frac{\partial K}{\partial \rho}\right)u^2\right] + 2k_2^2\frac{1-\rho}{\rho}K\frac{\partial q^{(2)}}{\partial \rho}\right\},$$

$$\alpha_2 = k_1^2\beta^2\rho(1-\rho)\left\{\frac{1}{\rho}K^2\frac{\partial q^{(1)}}{\partial \rho} + \right.$$

$$\left. + \left[K + (1-\rho)\frac{\partial K}{\partial \rho}\right]\left[\xi\varkappa K - (1 + \xi\varkappa)\rho\frac{\partial K}{\partial \rho}\right]u^2\right\} \qquad (1.7)$$

One-dimensional perturbations ($k_2 = 0$) are the most unstable if $(\partial/\partial\rho)q^{(2)} > 0$, while perturbations with large k_2 are so if the converse applies. If $(\partial/\partial\rho)q^{(i)} > 0$, loss of stability occurs first in respect of perturbations with $k_1 \approx 0$. The motion is unstable if $q^{(1)}$ decreases as ρ increases, because the roots of $(\partial/\partial\rho)q^{(1)} = 0$ (close to the concentrations ρ_* of a close-packed system [9, 10]) are usually much greater than the roots of

$$\xi\varkappa K = (1 + \xi\varkappa)\rho\partial K / \partial\rho.$$

It is therefore sufficient to examine the stability with respect to plane waves with $k_1 \approx 0$. Then (1.7) gives us an approximate stability condition:

$$\frac{1}{\rho}\frac{\partial q^{(1)}}{\partial \rho} - (1-\rho)\left(\frac{\partial \ln K}{\partial \rho} + \frac{1}{1-\rho}\right)\times$$

$$\times\left[(1 + \xi\varkappa)\rho\frac{\partial \ln K}{\partial \rho} - \xi\varkappa\right]u^2 > 0. \qquad (1.8)$$

It is clear that the stability is not dependent on the viscosities of the two phases.

§2. We need to know how K and $q^{(i)}$ vary with ρ and the other parameters in order to analyze (1.7) or (1.8). We can use for K the relation [9]

$$K \approx \frac{M}{(1-\rho)^{3.75}}[1 + 0.033(1-\rho)^{2.375}A^{1/2}],$$

$$A = \frac{8ga^3}{\mu_0^2}d_1(d_2 - d_1). \qquad (2.1)$$

It is difficult to derive expressions for the $q^{(i)}$ because the suspended particles move randomly in the viscous liquid; only approximate expressions are available, which are suitable when the characteristic time of interaction between the particles is much less than the relaxation time for the velocity of the particles relative to the liquid, while the interactions themselves are of short range [9, 10]. These conditions are usually obeyed by particles suspended in a gas, but only approximately for liquid suspensions.

It has been assumed [10] that a Markov process describes the motion of a system of particles in phase space, which gives the following expression for the normal stress of the dispersed phase in the direction of the flow:

$$P_{11} = -P_0 - \frac{4m\eta^\circ}{3\chi\theta}Bu^2, \quad P_0 = n\theta(1 + Y),$$

$$Y = 4\rho\chi, \qquad \eta^\circ = \frac{5}{64a^2}\left(\frac{m\theta}{\pi}\right)^{1/2}, \qquad B = \frac{3\beta K\theta}{mu^2},$$

$$\theta = \frac{D}{3}m\left(\frac{\partial \ln K}{\partial \rho} + \frac{1}{1-\rho}\right)^2\psi(\rho)u^2,$$

$$\psi(\rho) = \langle(\delta\rho)^2\rangle, \quad \chi = \chi(\rho). \qquad (2.2)$$

Here θ is the mean energy of the isotropic small-scale pulsations of the particles, B is a coefficient defining the diffusion tensor in pulsation-velocity space,

$\langle(\delta\rho)\rangle$ is the scale of the concentration fluctuations, D is a numerical factor left undefined in the theory of [10], and $\chi(\rho)$ indicates the increase in the frequency of binary collisions (all collisions are binary in a system of hard spheres) relative to that for a system of point particles. Enskog's theory of dense gases [10] gives the following result applicable to dilute systems:

$$\chi(\rho) \approx (1 - {}^{11}/_{2}\ \rho)\ (1 - 8\rho)^{-1}, \qquad \rho \ll 1.$$

For concentrated systems we have approximately that [9]

$$\chi(\rho) \approx \frac{1}{4\rho}\frac{\sigma}{1-\sigma}, \quad \sigma = \left(\frac{\rho}{\rho_*}\right)^{1/3}, \quad \sim 0.12 \leqslant \rho \leqslant \rho_*.$$

Also, $\chi(\rho)$ increases monotonically with ρ and tends to infinity as $\rho \to \rho_*$.

It follows from [9] that

$$\psi(\rho) \approx 6\rho^2 \frac{N\theta}{V}\left|\frac{1}{V}\left(\frac{\partial V}{\partial P_0}\right)_\theta\right| = 6\rho^2\left(1 + Y + \rho\frac{dY}{d\rho}\right)^{-1}. \tag{2.3}$$

For dilute and concentrated systems, respectively, we get

$$\psi(\rho) \approx 6\rho^2 \frac{1 - 16\rho + 64\rho^2}{1 - 8\rho + 2\rho^2} \text{ and } \psi(\rho) \approx 6\rho^2 \frac{(1-\sigma)^3}{1 - {}^{2}/_{3}\sigma}. \tag{2.4}$$

A formula has been derived [9] for θ that coincides with (2.2) apart from a numerical factor; comparison gives D ≈ 0.2, which value is used below.

Then we have

$$d_2 q^{(1)} = -P_{11} \approx 0.067\,\rho(1 + Y(\rho)) \times \psi(\rho)\left(\frac{\partial \ln K}{\partial\rho} + \frac{1}{1-\rho}\right)^2 d_2 u^2 + 0.573\frac{\psi^{1/2}(\rho)K}{\chi(\rho)}\left(\frac{\partial \ln K}{\partial\rho} + \frac{1}{1-\rho}\right)\left(\frac{\mu_0}{a d_2}\right) d_2 u. \tag{2.5}$$

We substitute (2.5) into (1.8) and write the condition as

$$2.57\,\Gamma\frac{\partial}{\partial\rho}\left[\frac{\psi^{1/2}K^2}{\chi}\left(\frac{\partial \ln K}{\partial\rho} + \frac{1}{1-\rho}\right)\right] + \left\{0.067\frac{\partial}{\partial\rho}\left[\rho\psi(1+Y)\left(\frac{\partial \ln K}{\partial\rho} + \frac{1}{1-\rho}\right)^2\right] - \rho(1-\rho)\left(\frac{\partial \ln K}{\partial\rho} + \frac{1}{1-\rho}\right)\left[(1+\xi\varkappa)\frac{\partial \ln K}{\partial\rho} - \xi\varkappa\right]\right\} > 0$$

$$\Gamma = \frac{\nu_0^2\varkappa^2}{(1-\varkappa)ga^3} = \frac{\mu_0^2}{(1-\varkappa)d_2^2 g a^3}, \quad A = \frac{8\varkappa}{\Gamma}, \quad u \neq 0 \tag{2.6}$$

The neutral curve is thus defined by ρ and the dimensionless parameter Γ, which is dependent on the parameters of the phases; this curve in the region $\Gamma > 0$, $0 \leq \rho \leq \rho_*$ is shown in Fig. 1 (the stability region lies above the curve). Instability sets in at a critical value $\Gamma_*(\rho)$ as Γ decreases, and this loss of stability corresponds to increase in particle size, decrease in the density ratio, increase in the density of the dispersed phase, and reduction in the viscosity of the dispersion medium. All these conclusions are in good general agreement with numerous experiments on flow in disperse systems and (especially) the fluidized state. Inhomogeneous behavior in a fluidized bed, for example, occurs more readily in a bed of large particles supported by a low-viscosity gas than it does in a bed of fine particles fluidized by a

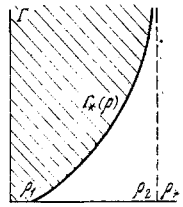

Fig. 1

liquid of high visocity and density (see [11] for review). Early turbulence in a dispersion is prevented by similar change in the phase parameters [5-7]. These effects can thus be related to loss of stability with respect to small perturbations.

If $\rho \leqslant \rho_2$ and Γ deviates slightly from the critical value (Fig. 1), instability occurs first in respect of a planar perturbation wave propagating along the flow direction, and it is possible for a secondary flow with $k_2 \approx 0$ to occur (stratification of a suspension). Conversely, $q^{(1)}(\rho)$ decreases as ρ increases for $\rho \gtrsim \rho_2$, and instability sets in with respect to perturbations having high wave numbers. This is precisely the region in which we expect rapid growth of initial perturbations up to the formation of bubbles filled with pure dispersion medium, e.g., as in an inhomogeneously fluidized bed.

These results correspond to the following physical picture of the behavior of perturbations. We assume, as an example, that a positive fluctuation occurs in the system (particle concentration above that in the surrounding medium). The upthrust on a particle from the fluid is larger than that on a particle outside the fluctuation [9], so the fluctuation begins to move upwards and traps fresh particles, thereby increasing in concentration. On the other hand, the interaction between the randomly moving particles results in an effective pressure in the solid; if this pressure increases with the concentration, the fluctuation will tend to disperse. These two opposing factors determine the stability

This explains why all previously published studies of linear stability in dispersed systems (e.g.,[12]) have led to the conclusion that such a system is unstable for arbitrary values of all physical parameters and of the concentration ρ, which obviously conflicts with experiment. The reason is that these studies made no allowance for the pressure in the solid as a function of ρ, and thus for the sole physical factor that tends to suppress concentration perturbations.

The value of ρ_1 (Fig. 1) is dependent on Γ and $\varkappa$, in general, because K in (2.1) is dependent on A, but this dependence is very weak.

We have $u \equiv 0$ for a system where the densities are equal, and the left sides in (1.7) and (1.8) are zero, which corresponds to neutral stability. The stability region changes substantially in character as u (and therefore Γ) decreases (becomes negative, particles less dense than the fluid). In particular, (1.7) shows that instability can occur for small ρ, whereas there is stability for large ρ. The first conclusion is in general agreement with experiment [7]. A more detailed study of the stability for $u < 0$ is made difficult by the fact that there is no proof that the formulas of §2 are applicable in this case, and no more accurate expressions are known for $q^{(i)}$ for such systems.

No account was taken [9,10] of momentum transfer by friction between solid particles in deducing the formulas of §2. It would be expected that this would cause $q^{(1)}(\rho)$ to increase with ρ in the region of ρ directly adjoining ρ_*, and so near ρ_* there should be an additional stability region. Experiment indicates that such a region exists (a fluidized bed near the onset of fluidization is homogeneous [11]), but it is presently unclear how it should be evaluated theoretically.

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